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Relativistic Neutron Spin-Echo Theory

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We consider the neutron Dirac equation with electric moment in addition to magnetic moment, solve it rigorously in a uniform electromagnetic field, and set up the relativistic neutron spin-echo theory with a magnetic moment. We also solve the equation in an alternating magnetic field.

1. INTRODUCTION

For the interaction between the electromagnetic field and the neutron's electric moment and magnetic moment, the total Lagrangian density is

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{em}} + \mathscr{L}_i \tag{1.1}$$

where

$$\mathscr{L}_0 = -\bar{\psi}(\gamma \cdot \partial + m)\psi$$

$$\mathscr{L}_{\text{em}} = -\frac{1}{2}(\partial_\mu A_\nu)^2$$

$$\mathscr{L}_i = \frac{\mu}{2}\bar{\psi}\sigma_{\mu\nu}\psi F_{\mu\nu} + \frac{\lambda}{2}\bar{\psi}\sigma_{\mu\nu}\psi f_{\mu\nu}$$

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

$$[F_{\mu\nu}] = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}$$

$$[f_{\mu\nu}] = \begin{bmatrix} 0 & E_3 & -E_2 & iB_1 \\ -E_3 & 0 & E_1 & iB_2 \\ E_2 & -E_1 & 0 & iB_3 \\ -iB_1 & -iB_2 & -iB_3 & 0 \end{bmatrix}$$

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Here μ and λ are, respectively, the magnetic moment and the electric moment of the particles. $F_{\mu\nu}$ is the electromagnetic field tensor and $f_{\mu\nu}$ is the dual tensor of $F_{\mu\nu}$. From the Euler–Lagrange equation we can obtain the neutron Dirac equation

$$\left(\gamma\cdot\partial+m-\frac{\mu}{2}\sigma_{\mu\nu}F_{\mu\nu}-\frac{\lambda}{2}\sigma_{\mu\nu}f_{\mu\nu}\right)\psi=0 \tag{1.2}$$

For $\lambda=0$, equation (1.2) becomes the usual result (Zhang, 1989)

$$\left(\gamma\cdot\partial+m-\frac{\mu}{2}\sigma_{\mu\nu}F_{\mu\nu}\right)\psi=0 \tag{1.3}$$

It is easy to prove that there exists a special $U(1)$ symmetry group. We denote

$$G_{\mu\nu}=\begin{pmatrix}F_{\mu\nu}\\ f_{\mu\nu}\end{pmatrix},\qquad D=\begin{pmatrix}\mu\\ \lambda\end{pmatrix} \tag{1.4}$$

Then we obtain

$$\left(\gamma\cdot\partial+m-\frac{1}{2}\sigma_{\mu\nu}\tilde{D}G_{\mu\nu}\right)\psi=0 \tag{1.5}$$

We apply the $U(1)$ rotation of the following form to all two-component vectors $\binom{a}{b}$:

$$\begin{pmatrix}a\\ b\end{pmatrix}\to\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} \tag{1.6}$$

We may find that equation (1.5) does not change under the transformation (1.6). But this symmetry does not introduce a Noether current.

Now we consider a general Dirac equation

$$i\frac{\partial}{\partial t}\psi=(\boldsymbol{\alpha}\cdot\mathbf{P}+\beta m+H_I)\psi \tag{1.7}$$

where H_I is a bounded Hermitian operator without any differential operator in it. It is regular everywhere except the boundary surface, and not continuous on the boundary surface. It can be proved that the four components of ψ are continuous on the boundary by taking a small cylinder through the boundary surface.

Lemma. Suppose that $\hat{M}$, $\hat{N}$ are two commuting Hermitian operators, and they have the same orthogonal and normalized complete set of states $\{|m_i, n_j, l_k\rangle\}$. (Here l_k are some eigenvalues of other operators in the complete commuting set of operators.) We have

$$\hat{M}|m_i, n_j, l_k\rangle = m_i|m_i, n_j, l_k\rangle$$

$$\hat{N}|m_i, n_j, l_k\rangle = n_j|m_i, n_j, l_k\rangle$$

Then the solutions $|\varphi\rangle$ of the equation $\hat{M}|\varphi\rangle = \hat{N}|\varphi\rangle$ are only composed of states in which m_i is equal to n_j.

Proof omitted.

2. RIGOROUS SOLUTION FOR NEUTRON DIRAC EQUATION IN A UNIFORM ELECTROMAGNETIC FIELD

We apply transformation (1.6) to (1.2), and choose θ such that $\sin\theta = \lambda/(\lambda^2+\mu^2)^{1/2}$ and $\cos\theta = \mu/(\lambda^2+\mu^2)^{1/2}$. Equation (1.2) takes the form,

$$\left(\gamma\cdot\partial + m - \frac{\mu'}{2}\sigma_{\mu\nu}F'_{\mu\nu}\right)\psi = 0 \tag{2.1}$$

where

$$\begin{aligned} \mu' &= \mu\cos\theta + \lambda\sin\theta, \qquad & \lambda' &= -\mu\sin\theta + \lambda\cos\theta \\ F'_{\mu\nu} &= F_{\mu\nu}\cos\theta + f_{\mu\nu}\sin\theta, \qquad & f'_{\mu\nu} &= -F_{\mu\nu}\sin\theta + f_{\mu\nu}\cos\theta \end{aligned} \tag{2.2}$$

$F'_{\mu\nu}$ is the effective electromagnetic field tensor, so the "elimination" of λ is only in form, not in essence.

For convenience, we omit the prime in the following calculations. Suppose the stationary state wave function of equation (2.1) to be

$$\psi(\mathbf{r}, t) = \begin{pmatrix}\phi(\mathbf{r})\\ \chi(\mathbf{r})\end{pmatrix} e^{-i\varepsilon t} \tag{2.3}$$

Substituting this into equation (2.1), we get

$$\begin{aligned} &\{\varepsilon - m - [(\varepsilon+m)^2 - \mu^2\mathbf{B}^2]^{-1}[(\varepsilon+m)(\mathbf{P}^2+\mu^2\mathbf{E}^2) - 2\mathbf{P}\cdot(\mathbf{E}\times\mathbf{B})\mu^2]\}\phi \\ &\quad = \{[(\varepsilon+m)^2-\mu^2\mathbf{B}^2]^{-1}[2(\varepsilon+m)\mathbf{P}\times\mathbf{E}^{\mu} + 2\mu(\mathbf{P}\cdot\mathbf{B})\mathbf{P} + 2\mu^3(\mathbf{B}\cdot\mathbf{E})\mathbf{E} \\ &\qquad - \mu(\mathbf{P}^2+\mu^2\mathbf{E}^2)\mathbf{B}] - \mu\mathbf{B}\}\cdot\boldsymbol{\sigma}\phi \end{aligned} \tag{2.4a}$$

$$\chi = \frac{(\varepsilon + m + \mu\boldsymbol{\sigma}\cdot\mathbf{B})\boldsymbol{\sigma}\cdot(\mathbf{P}-i\mu\mathbf{E})}{(\varepsilon+m)^2-\mu^2\mathbf{B}^2}\phi \tag{2.4b}$$

It is easy to find that equation (2.4a) has the form $\hat{M}\phi = \hat{N}\phi$, where $\hat{N} = \mathbf{V}\cdot\boldsymbol{\sigma}$, and

$$\begin{aligned} \hat{M}(\mathbf{P},\mathbf{E},\mathbf{B},\varepsilon) &= \varepsilon - m - [(\varepsilon+m)^2-\mu^2\mathbf{B}^2]^{-1}[(\varepsilon+m)(\mathbf{P}^2+\mu^2\mathbf{E}^2) \\ &\qquad - 2\mathbf{P}\cdot(\mathbf{E}\times\mathbf{B})\mu^2] \\ \mathbf{V}(\mathbf{P},\mathbf{E},\mathbf{B},\varepsilon) &= [(\varepsilon+m)^2-\mu^2\mathbf{B}^2]^{-1}[2(\varepsilon+m)\mathbf{P}\times\mathbf{E}\mu + 2\mu(\mathbf{P}\cdot\mathbf{B})\mathbf{P} \\ &\qquad + 2\mu^3(\mathbf{B}\cdot\mathbf{E})\mathbf{E} - \mu(\mathbf{P}^2+\mu^2\mathbf{E}^2)\mathbf{B}] - \mu\mathbf{B} \end{aligned} \tag{2.5}$$

According to the Lemma, we know that $\phi(\mathbf{r})$ is composed of such common eigenstates of $\hat{M}$ and $\hat{N}$ in which $m = n$. Obviously, the eigenstate of $\hat{M}$ is also the eigenstate of the momentum operator $\mathbf{P}$; we denote it by $U \exp(i\mathbf{P} \cdot \mathbf{r})$. Here U is a two-component spinor. Substituting it into the equation $\hat{M}\phi = \hat{N}\phi$, we find

$$\mathbf{V} \cdot \boldsymbol{\sigma} U = nU \tag{2.6}$$

The solutions are

$$U(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) = c\begin{pmatrix} V_x(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) - iV_y(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) \\ sV(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) - V_z(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) \end{pmatrix} \tag{2.7a}$$

$$\begin{aligned} n &= sV(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) \\ &= s[V_x^2(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) + V_y^2(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) + V_z^2(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon)]^{1/2} \end{aligned} \tag{2.7b}$$

Here c is a normalization constant, and $s = \pm 1$ represents the direction of spin. So we affirm the common eigenstate of $\hat{M}$ and $\hat{N}$ to be

$$\phi(\mathbf{r}) = c\begin{pmatrix} V_x(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) - iV_y(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) \\ sV(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) - V_z(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) \end{pmatrix} e^{i\mathbf{P} \cdot \mathbf{r}} \tag{2.8}$$

From the Lemma, the following equation must hold:

$$m(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) = sv(\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon) \tag{2.9}$$

Here v is obtained by substituting the corresponding eigenvalue of $\mathbf{P}$ into the formula of V. Substituting an arbitrary set of $\{\mathbf{P}, \mathbf{E}, \mathbf{B}, \varepsilon\}$ which satisfies (2.9) into (2.8), we get a rigorous solution of (2.4a); further, we find the corresponding solution of (2.4b). Thus (2.1) is solved.

In the following we try to find the energy spectrum ε.

Multiplying (1.3) by the operator $(\gamma \cdot \partial + m + 1/2\mu\sigma_{\mu\nu}F_{\mu\nu})$, we get

$$\begin{aligned} &\left(m^2 - P^2 - \frac{\mu^2}{2} F_{\mu\nu}F_{\mu\nu}\right)^2 \psi \\ &= \left[\frac{i}{2}\mu^2 F_{\mu\nu}f_{\mu\nu}\gamma_5 - (2imP_\mu - 2\mu P_\nu F_{\nu\mu})\gamma_\mu\right]^2 \psi \\ &= \left[-\frac{\mu^4}{4}(F_{\mu\nu}f_{\mu\nu})^2 - 4m^2P^2 + 4\mu^2(P_\mu F_{\mu\nu})^2\right]\psi \end{aligned} \tag{2.10}$$

and after using

$$\begin{aligned} F_{\mu\nu}f_{\mu\nu} &= 4\mathbf{E} \cdot \mathbf{B} \\ F_{\mu\nu}F_{\mu\nu} &= 2(\mathbf{B}^2 - \mathbf{E}^2) \\ (P_\mu F_{\mu\nu})^2 &= (\mathbf{B} \times \mathbf{P})^2 + \varepsilon^2\mathbf{E}^2 + 2\varepsilon\mathbf{P} \cdot (\mathbf{B} \times \mathbf{E}) - (\mathbf{P} \cdot \mathbf{E})^2 \end{aligned}$$

we find that equation (2.10) becomes

$$(\varepsilon^2 + m^2 - \mathbf{P}^2 + \mu^2\mathbf{E}^2 - \mu^2\mathbf{B}^2)^2\psi$$
$$= 4\{\mu^2[(\mathbf{B}\times\mathbf{P})^2 + \varepsilon^2\mathbf{E}^2 + 2\varepsilon\mathbf{P}\cdot(\mathbf{B}\times\mathbf{E}) - (\mathbf{P}\cdot\mathbf{E})^2]$$
$$- m^2\mathbf{P}^2 + m^2\varepsilon^2 - \mu^4(\mathbf{E}\cdot\mathbf{B})^2\}\psi$$

Because ψ is arbitrary, we may remove it from both sides; then we obtain the equation of the energy spectrum as follows:

$$(\varepsilon^2 - \mathbf{P}^2 - m^2 - \mu^2\mathbf{B}^2 - \mu^2\mathbf{E}^2)^2 - 8\varepsilon\mathbf{P}$$
$$\cdot(\mathbf{B}\times\mathbf{E})\mu^2 - 4[(\mu\mathbf{B}\times\mathbf{P})^2 + (\mathbf{P}\times\mu\mathbf{E})^2$$
$$+ (\mu\mathbf{E}\times\mu\mathbf{B})^2 + m^2\mu^2\mathbf{B}^2] = 0 \tag{2.11}$$

This is a fourth-power equation; we may get ε by solving the equation. In particular, if $\mathbf{P}\cdot(\mathbf{B}\times\mathbf{E}) = 0$, i.e., $\mathbf{P}$, $\mathbf{B}$, $\mathbf{E}$ are in one plane, (2.11) may be turned into a second-power equation as follows:

$$\varepsilon^2 - \mathbf{P}^2 - m^2 - \mu^2\mathbf{B}^2 - \mu^2\mathbf{E}^2$$
$$= \pm 2[(\mu\mathbf{B}\times\mathbf{P})^2 + (\mathbf{P}\times\mu\mathbf{E})^2 + (\mu\mathbf{E}\times\mu\mathbf{B})^2 + m^2\mu^2\mathbf{B}^2]^{1/2} \tag{2.12}$$

When the electric moment exists, we denote

$$\mu = d\cos\theta$$
$$\lambda = d\sin\theta$$
$$d = (\mu^2 + \lambda^2)^{1/2}$$

and the energy spectrum equation takes the form

$$[\varepsilon^2 - \mathbf{P}^2 - m^2 - d^2(\mathbf{B}^2 + \mathbf{E}^2)]^2 - 4d^2m^2$$
$$\times(\mathbf{B}\cos\theta + \mathbf{E}\sin\theta)^2 - 4d^2[(\mathbf{E}\times\mathbf{B})^2 d^2$$
$$+ (\mathbf{B}\times\mathbf{P})^2 + (\mathbf{P}\times\mathbf{E})^2] + 8\varepsilon d^2\mathbf{P}\cdot(\mathbf{E}\times\mathbf{B}) = 0 \tag{2.13}$$

It is noticeable that when $8\varepsilon d^2\mathbf{P}\cdot(\mathbf{E}\times\mathbf{B}) \neq 0$, the positive and negative energy levels are unsymmetric. For the same energy level, if $\mathbf{E}$ and $\mathbf{B}$ are fixed, ε is a function of both the momentum $\mathbf{P}$ and the direction of spin. When we only consider the positive energy solutions, there are two and respectively correspond to $s = +1, s = -1$,

$$\varepsilon = \varepsilon(\mathbf{P}, \mathbf{E}, \mathbf{B}, s) \tag{2.14}$$

From all of the above, the general solution of equation (2.1) in a uniform electromagnetic field is

$$\psi(\mathbf{r}, t) = \sum_{\mathbf{P}} \sum_{s=\pm 1} \sum_{\varepsilon=\varepsilon(\mathbf{P},\mathbf{E},\mathbf{B},s)} \times \left\{ A(\mathbf{P}, s, \varepsilon) \cdot \begin{pmatrix} I & 0 \\ 0 & \dfrac{\boldsymbol{\sigma} \cdot (\mathbf{P} - i\mu \mathbf{E})}{\varepsilon + m - \mu \boldsymbol{\sigma} \cdot \mathbf{B}} \end{pmatrix} \cdot \begin{bmatrix} V_x - iV_y \\ sV - V_z \\ V_x - iV_y \\ sV - V_z \end{bmatrix} \cdot e^{i(\mathbf{P} \cdot \mathbf{r} - \varepsilon t)} \right\} \tag{2.15}$$

The coefficients $A(\mathbf{P}, s, \varepsilon)$ are determined by the normalization conditions and the boundary conditions.

If we only consider neutrons moving along $\pm z$ directions, the wave functions are four-fold degenerate. We denote these wave functions by (omitting the normalization constants)

$$\phi = \begin{cases} \phi(\mathbf{P} = \mathbf{P}^{+}_{s=+1}, s = +1) \equiv \phi^{++}(z) \\ \phi(\mathbf{P} = \mathbf{P}^{+}_{s=-1}, s = -1) \equiv \phi^{+-}(z) \\ \phi(\mathbf{P} = \mathbf{P}^{-}_{s=+1}, s = +1) \equiv \phi^{-+}(z) \\ \phi(\mathbf{P} = \mathbf{P}^{-}_{s=-1}, s = -1) \equiv \phi^{--}(z) \end{cases} \tag{2.16}$$

where

$$\mathbf{P} = \begin{cases} \mathbf{P}^{+} \equiv (0, 0, \mathbf{P}^{+}(\varepsilon, \mathbf{B}, \mathbf{E}, s)), & \mathbf{P}^{+}(\varepsilon, \mathbf{B}, \mathbf{E}, s) > 0, \quad \mathbf{P} \text{ in } +z \text{ direction} \\ \mathbf{P}^{-} \equiv (0, 0, -\mathbf{P}^{-}(\varepsilon, \mathbf{B}, \mathbf{E}, s)), & \mathbf{P}^{-}(\varepsilon, \mathbf{B}, \mathbf{E}, s) > 0, \quad \mathbf{P} \text{ in } -z \text{ direction} \end{cases}$$

$\chi(z)$ can be obtained similarly.

3. RELATIVISTIC NEUTRON SPIN-ECHO THEORY

Mezei (1972) discussed the idea of the neutron spin echo (Fig. 1). But Mezei's theory is based on the Pauli equation, and is doubtful for the relativistic case. Here we discuss the problem based on the rigorous solution of equation (2.1).

It is known that the neutron wave function in every zone can be expanded with $\{\phi^{++}, \phi^{+-}, \phi^{-+}, \phi^{--}\}$ and $\{\chi^{++}, \chi^{+-}, \chi^{-+}, \chi^{--}\}$.

$$\phi_{\mathrm{I}}(z) = A_{\mathrm{I}}\phi^{++} + B_{\mathrm{I}}\phi^{+-} + C_{\mathrm{I}}\phi^{-+} + D_{\mathrm{I}}\phi^{--} \tag{3.1a}$$

$$\phi_{\mathrm{II}}(z) = A_{\mathrm{II}}\phi^{++} + B_{\mathrm{II}}\phi^{+-} + C_{\mathrm{II}}\phi^{-+} + D_{\mathrm{II}}\phi^{--} \tag{3.1b}$$

$$\phi_{\mathrm{III}}(z) = A_{\mathrm{III}}\phi^{++} + B_{\mathrm{III}}\phi^{+-} + C_{\mathrm{III}}\phi^{-+} + D_{\mathrm{III}}\phi^{--} \tag{3.1c}$$

$$\phi_{\mathrm{IV}}(z) = A_{\mathrm{IV}}\phi^{++} + B_{\mathrm{IV}}\phi^{+-} \tag{3.1d}$$

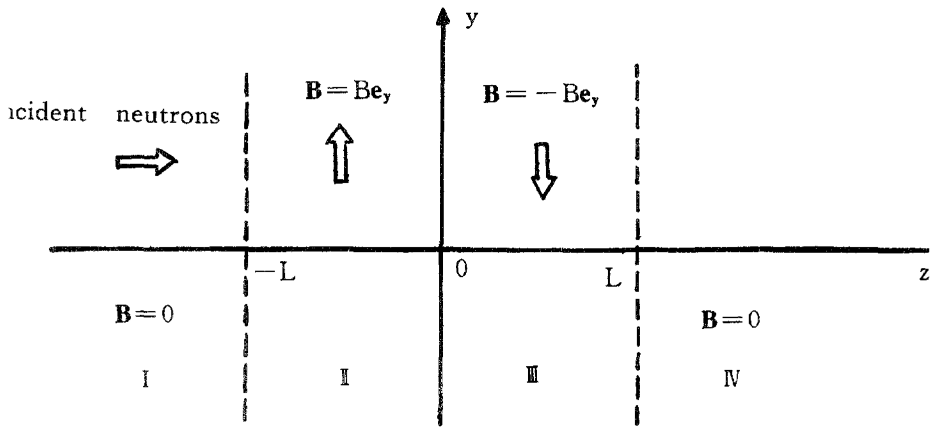

Fig. 1

From equation (2.5) we obtain

$$V_x = V_z = 0, \qquad V_y = -\mu B_y\left(1 + \frac{P^2}{(\varepsilon + m)^2 - \mu^2\mathbf{B}^2}\right) \tag{3.2}$$

The eigenstates of $\mathbf{V}\cdot\boldsymbol{\sigma}$ are (omitting the normalization constants)

$$\alpha_s = \begin{pmatrix} iB_y \\ sB \end{pmatrix}, \qquad s = \pm 1 \tag{3.3}$$

Then we find

$$\phi_{\rm I} = \phi_{\rm I}^s \equiv e^{iPz}\alpha_s + re^{-iPz}\alpha_s \tag{3.4a}$$

$$\phi_{\rm II} = \phi_{\rm II}^s \equiv ae^{iP_s^+ z}\alpha_s + be^{-iP_s^- z}\alpha_s \tag{3.4b}$$

$$\phi_{\rm III} = \phi_{\rm III}^s \equiv ce^{iP^+_{-s} z}\alpha_s + de^{-iP^-_{-s} z}\alpha_s \tag{3.4c}$$

$$\phi_{\rm IV} = \phi_{\rm IV}^s \equiv te^{iPz}\alpha_s \tag{3.4d}$$

Similarly we get

$$\chi_{\rm I} = \frac{P}{\varepsilon + m}e^{iPz}\alpha_{-s} - r\frac{P}{\varepsilon + m}e^{-iPz}\alpha_{-s} \tag{3.5a}$$

$$\chi_{\rm II} = \frac{aP_s^+}{\varepsilon + m - s\mu B}e^{iP_s^+ z}\alpha_{-s} - \frac{bP_s^-}{\varepsilon + m - s\mu B}e^{-iP_s^- z}\alpha_{-s} \tag{3.5b}$$

$$\chi_{\rm III} = \frac{cP^+_{-s}}{\varepsilon + m + s\mu B}e^{iP^+_{-s} z}\alpha_{-s} - \frac{dP^-_{-s}}{\varepsilon + m + s\mu B}e^{-iP^-_{-s} z}\alpha_{-s} \tag{3.5c}$$

$$\chi_{\rm IV} = t\,\frac{P}{\varepsilon + m}\,e^{iPz}\alpha_{-s} \tag{3.5d}$$

Here r, a, b, c, d, and t are six constant coefficients. It is easy to prove that $P_s^+ = P_s^-$, and we denote them as P_s; therefore, we have

$$P_s = (\varepsilon^2 - m^2 + \mu^2\mathbf{B}^2 + 2s\varepsilon\mu B)^{1/2} \tag{3.6}$$

From the continuous conditions

$$\phi_{\rm I}(-L) = \phi_{\rm II}(-L), \qquad \phi_{\rm II}(0) = \phi_{\rm III}(0), \qquad \phi_{\rm III}(L) = \phi_{\rm IV}(L)$$
$$\chi_{\rm I}(-L) = \chi_{\rm II}(-L), \qquad \chi_{\rm II}(0) = \chi_{\rm III}(0), \qquad \chi_{\rm III}(L) = \chi_{\rm IV}(L)$$

we may calculate t. Denoting $A_s = P_s/(\varepsilon + m - \mu s B)$, $A = P/(\varepsilon + m)$, $l = e^{-iPL}$, and $l_s = e^{-iP_sL}$, we find

$$\begin{aligned} t = 2A\Bigg[&\left(\frac{A}{2} + \frac{A^2}{4A_{-s}} + \frac{AA_{-s}}{4A_s} + \frac{A^2}{4A_s} + \frac{A_s}{4} + \frac{AA_s}{4A_{-s}} + \frac{A_{-s}}{4}\right) l^2 l_s^{-1} l_{-s}^{-1} \\ &+ \left(\frac{A}{2} - \frac{A_{-s}}{4} - \frac{A_s}{4} - \frac{A^2}{4A_{-s}} - \frac{A^2}{4A_s} + \frac{AA_{-s}}{4A_s} + \frac{AA_s}{4A_{-s}}\right) l^2 l_s l_{-s} \\ &+ \left(\frac{A}{2} + \frac{A^2}{4A_s} - \frac{A^2}{4A_s} - \frac{AA_{-s}}{4A_{-s}} + \frac{A_s}{4} - \frac{AA_s}{4A_{-s}} - \frac{A_{-s}}{4}\right) l^2 l_{-s} l_s^{-1} \\ &+ \left(\frac{A}{2} + \frac{A^2}{4A_{-s}} - \frac{A^2}{4A_s} - \frac{AA_s}{4A_{-s}} + \frac{A_{-s}}{4} - \frac{AA_{-s}}{4A_s} - \frac{A_s}{4}\right) l^2 l_s l_{-s}^{-1} \Bigg]^{-1} \end{aligned} \tag{3.7a}$$

$$\begin{aligned} r = \frac{t}{4}\Bigg[& l_s^{-1} l_{-s}^{-1} + \frac{A}{A_{-s}} l_s^{-1} l_{-s}^{-1} + \frac{A}{A_s} l_s^{-1} l_{-s}^{-1} + l_s^{-1} l_{-s} + l_s l_{-s}^{-1} - \frac{A}{A_{-s}} l_s^{-1} l_{-s} \\ &- \frac{A}{A_s} l_s l_{-s}^{-1} + \frac{A_{-s}}{A_s} l_s^{-1} l_{-s}^{-1} - \frac{A_s}{A_{-s}} l_s^{-1} l_{-s} - \frac{A_{-s}}{A_s} l_{-s}^{-1} l_s - \frac{A}{A_{-s}} l_s l_{-s} \\ &- \frac{A}{A_s} l_s l_{-s} + l_s l_{-s} + \frac{A}{A_s} l_s^{-1} l_{-s} + \frac{A}{A_{-s}} l_s l_{-s}^{-1} + \frac{A_s}{A_{-s}} l_s^{-1} l_s \Bigg] - l^2 \end{aligned} \tag{3.7b}$$

Obviously, in equations (3.7), s and $-s$ are symmetric, i.e., incident neutrons with different spin directions have the same reflection and transmission coefficients.

Suppose the incident wave function to be $\phi_i = k_1\phi^{s=+1} + k_2\phi^{s=-1}$, where k_1 and k_2 are two superposition coefficients. The reflection and transmission waves possess the same ratio of superposition coefficients as that of the incident wave. That is to say, the neutron state does not change after passing the four zones. The second magnetic field "eliminates" the effect of the first magnetic field just as in the nonrelativistic case.

Now we calculate the neutron current $j_\mu = c\bar{\psi}\gamma_\mu\psi$. We suppose the neutron wave functions in zones I and IV to be

$$\psi_{\mathrm{I}}(z) = e^{iPz}\begin{bmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\\ \dfrac{P\sigma_3}{\varepsilon+m}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\end{bmatrix} + re^{-iPz}\begin{bmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\\ \dfrac{-P\sigma_3}{\varepsilon+m}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\end{bmatrix} \tag{3.8a}$$

$$\psi_{\mathrm{IV}}(z) = te^{iPz}\begin{bmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\\ \dfrac{P\sigma_3}{\varepsilon+m}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\end{bmatrix} \tag{3.8b}$$

where $\binom{\alpha}{\beta}$ is an arbitrary two-component spinor. We get

$$J_z^{\mathrm{I}} = (|\alpha|^2+|\beta|^2)\frac{2P}{\varepsilon+m}(1-|r|^2), \qquad J_x^{\mathrm{I}} = J_y^{\mathrm{I}} = 0 \tag{3.9a}$$

$$J_z^{\mathrm{IV}} = (|\alpha|^2+|\beta|^2)\frac{2P}{\varepsilon+m}|t|^2, \qquad J_x^{\mathrm{IV}} = J_y^{\mathrm{IV}} = 0 \tag{3.9b}$$

Because the current is conservative, we obtain

$$|r|^2+|t|^2 = 1 \tag{3.10}$$

Generally, even if the magnetic field is very strong, we can also adjust the other conditions so that $|t|$ can be equal to 1 and $|r|$ is equal to zero, i.e., neutrons are not reflected by the magnetic field. The resonance conditions for transmission are

$$\begin{aligned}\mathbf{P}(s=+1,\varepsilon,\mathbf{B})\cdot L &= n\pi\\ \mathbf{P}(s=-1,\varepsilon,\mathbf{B})\cdot L &= n'\pi\end{aligned} \qquad (n, n' \text{ are integers}) \tag{3.11}$$

Under these conditions, we have

$$\begin{aligned} t &= e^{-2i(\varepsilon^2-m^2)^{1/2}L}e^{i(n-n')\pi}\\ r &= 0\end{aligned} \tag{3.12}$$

Now we have established the spin-echo theory for the relativistic neutron with magnetic moment, and pointed out that Mezei's theory may be extended to the relativistic case.

4. NEUTRON DIRAC EQUATION IN AN ALTERNATING MAGNETIC FIELD

Now we consider the case that the neutrons pass a zone with an alternating magnetic field (Fig. 2). Assume that $\mathbf{P} = (0, P, 0)$, and suppose

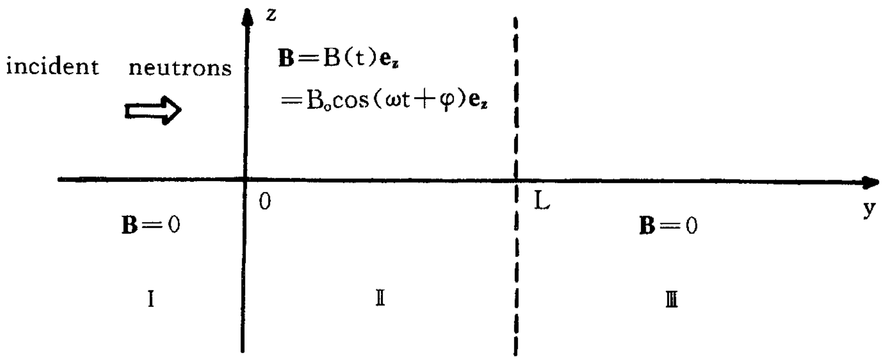

Fig. 2

the wavefunction in $0 < y < L$ to be

$$\psi_{\rm II}(y) = \begin{pmatrix} \phi_1(t) \\ \chi_1(t) \end{pmatrix} e^{iPy} + \begin{pmatrix} \phi_2(t) \\ \chi_2(t) \end{pmatrix} e^{-iPy} \tag{4.1}$$

After substituting equation (4.1) into (1.3), we easily get

$$i\begin{pmatrix} \dot{\phi}_1(t) \\ \dot{\chi}_1(t) \end{pmatrix} = \begin{pmatrix} m - \mu\sigma_3 B(t) & \sigma_2 P \\ \sigma_2 P & -m + \mu\sigma_3 B(t) \end{pmatrix} \begin{pmatrix} \phi_1(t) \\ \chi_1(t) \end{pmatrix} \tag{4.2a}$$

$$i\begin{pmatrix} \dot{\phi}_2(t) \\ \dot{\chi}_2(t) \end{pmatrix} = \begin{pmatrix} m - \mu\sigma_3 B(t) & -\sigma_2 P \\ -\sigma_2 P & -m + \mu\sigma_3 B(t) \end{pmatrix} \begin{pmatrix} \phi_2(t) \\ \chi_2(t) \end{pmatrix} \tag{4.2b}$$

From equation (4.2a) we find

$$\chi_1(t) = \frac{\sigma_2}{P}[i\dot{\phi}_1(t) - (m - \mu\sigma_3 B)\phi_1(t)] \tag{4.3a}$$

$$\phi_1(t) = \frac{\sigma_2}{P}[i\dot{\chi}_1(t) + (m - \mu\sigma_3 B)\chi_1(t)] \tag{4.3b}$$

We substitute equation (4.3a) into (4.3b) and obtain

$$\ddot{\phi}_1 - 2i\mu\sigma_3 B\dot{\phi}_1 + (P^2 + m^2)\phi_1 - (i\mu\sigma_3\dot{B} + \mu^2 B^2)\phi_1 = 0 \tag{4.4}$$

Denoting $\phi_1(t) = \phi'_1(t)e^{i\varepsilon' t}$ and $\varepsilon' = (P^2 + m^2)^{1/2}$, we get

$$\ddot{\phi}'_1 - 2i(\varepsilon' + \mu\sigma_3 B)\dot{\phi}'_1 - (i\mu\sigma_3\dot{B} + 2\mu\sigma_3 B\varepsilon' + \mu^2 B^2)\phi'_1 = 0 \tag{4.5}$$

Let

$$\phi'_1(t) = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}$$

Then equation (4.5) becomes

$$\begin{aligned} &\ddot{g} - 2i(\varepsilon' + \mu B)\dot{g} - (i\mu\dot{B} + 2\mu B\varepsilon' + \mu^2 B^2)g = 0 \\ &\ddot{h} - 2i(\varepsilon' - \mu B)\dot{h} + (i\mu\dot{B} + 2\mu B\varepsilon' - \mu^2 B^2)h = 0 \end{aligned} \tag{4.6}$$

The solutions of (4.6) are

$$\begin{aligned} g(t) &= e^{[i\mu B_0 \sin(\omega t + \varphi)]/\omega}(c_1 + c_2 e^{2i\varepsilon' t}) \\ h(t) &= e^{-[i\mu B_0 \sin(\omega t + \varphi)]/\omega}(c_1' + c_2' e^{2i\varepsilon' t}) \end{aligned} \tag{4.7}$$

Thus,

$$\phi_1(t) \equiv \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} e^{[i\mu B_0 \sin(\omega t + \varphi)]/\omega}(c_1 e^{-i\varepsilon' t} + c_2 e^{i\varepsilon' t}) \\ e^{-[i\mu B_0 \sin(\omega t + \varphi)]/\omega}(c_1' e^{-i\varepsilon' t} + c_2' e^{i\varepsilon' t}) \end{pmatrix} \tag{4.8a}$$

$$\chi_1(t) = \frac{1}{P}\begin{pmatrix} \dot{\eta} + im\eta - i\mu B_0 \cos(\omega t + \varphi)\eta \\ -\xi - im\xi + i\mu B_0 \cos(\omega t + \varphi)\xi \end{pmatrix} \tag{4.8b}$$

The solution of equation (4.2b) can be obtained similarly.

Now we only consider the positive energy states; the ϕ in three zones are

$$\phi_{\rm I} = \begin{pmatrix} A \\ A' \end{pmatrix} e^{iPy - i\varepsilon' t} + r\begin{pmatrix} A \\ A' \end{pmatrix} e^{-iPy - i\varepsilon' t} \tag{4.9a}$$

$$\phi_{\rm II} = \begin{pmatrix} Ce^{[i\mu B_0 \sin(\omega t + \varphi)]/\omega} \\ C'e^{-[i\mu B_0 \sin(\omega t + \varphi)]/\omega} \end{pmatrix} e^{iPy - i\varepsilon' t} + \begin{pmatrix} De^{[i\mu B_0 \sin(\omega t + \varphi)]/\omega} \\ D'e^{-[i\mu B_0 \sin(\omega t + \varphi)]/\omega} \end{pmatrix} e^{-iPy - i\varepsilon' t} \tag{4.9b}$$

$$\phi_{\rm III} = \begin{pmatrix} F \\ F' \end{pmatrix} e^{iPy - i\varepsilon' t} \tag{4.9c}$$

where A, A', C, C', D, D', F, F' are eight constant coefficients. Here we omitted the expressions of χ.

We suppose that the neutrons reach $y = 0$ as $t = 0$ and $y = L$ as $t = T$. From the boundary conditions we obtain

$$\begin{aligned} C &= e^{-(i\mu B_0/\omega)\sin\varphi} A \\ C' &= e^{(i\mu B_0/\omega)\sin\varphi} A' \\ D &= re^{-(i\mu B_0/\omega)\sin\varphi} A \\ D' &= re^{(i\mu B_0/\omega)\sin\varphi} A' \\ F &= e^{(i\mu B_0/\omega)[\sin(\omega T + \varphi) - \sin\varphi]}(1 + re^{-iPL})A \\ F' &= e^{-(i\mu B_0/\omega)[\sin(\omega T + \varphi) - \sin\varphi]}(1 + re^{-iPL})A' \end{aligned} \tag{4.10}$$

Now we try to find the resonance condition. It requires that

$$\phi_{\rm III} \propto \begin{pmatrix} A \\ A' \end{pmatrix} e^{iPy - i\varepsilon' t}$$

i.e., $F/A = F'/A'$; then we obtain

$$\frac{\mu B_0}{\omega}[\sin(\omega T + \varphi) - \sin \varphi] = n\pi \tag{4.11}$$

where n is an arbitrary integer. If $\varphi = 0$, equation (4.11) becomes

$$\frac{\mu B_0}{\omega} \sin \omega T = n\pi \tag{4.12}$$

If the above condition is satisfied, the effect of the magnetic field seems to be "eliminated." As a whole, the zone of the magnetic field is like a free space.

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